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(Residential Autonomous Degree College with P.G. Section under University of Calcutta)

B.A./B.SC. SECOND SEMESTER EXAMINATION, MAY 2011

FIRST YEAR

MATHEMATICS (Honours)

Date : 24/05/2011

Time : 11am – 3pm

Paper : II

Full Marks : 100

[Use separate Answer Scripts for each group]

Group – A

Answer any five questions from Q. No. 1 to Q. No. 8 :

1. If a_1, a_2, …, a_n be n positive real numbers then prove that $\frac{a_1+a_2+\ldots+a_n}{n} \geq \sqrt[n]{(a_1a_2\ldots a_n)}$ and equality occurs when $a_1 = a_2 = \ldots = a_n$ [5]

2. a) If $S_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$, prove that $n+S_n > n(n+1)^{1/n}$ if $n > 1$.
   b) Find the greatest value of $(x+2)^5 (7-x)^4$ when $-2 < x < 7$. Also find the value of x for which the greatest value is attained. [2+3]

3. a) Find the principal argument of $1+\cos 2\theta + i\sin 2\theta$, $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$
   b) Find all values of $(-i)^{1/4}$. [3+2]

4. a) If Z is a non-zero complex number and m be a positive integer, prove that $\operatorname{Log} Z^{1/m} = \frac{1}{m}\operatorname{Log} Z$.
   b) Solve $\exp Z = 1+\sqrt{3}\,i$. [3+2]

5. Solve $x^4 + 2x^3 - 18x^2 + 6x + 9 = 0$ given that the ratio of two of its roots is equal to the ratio of the other two. [5]

6. Obtain the equation whose roots are the roots of the equation $4x^3 - 8x^2 - 19x + 26 = 0$ each diminished by 2. Use Descartes' rule of signs to both the equations to find the exact number of positive and negative roots of the given equation. [2+3]

7. Define special root of $x^n - 1 = 0$, $n > 1$.
   If α be a special root of $x^{12} - 1 = 0$, prove that $(\alpha+\alpha^{11})(\alpha^5+\alpha^7) = -3$. [1+4]

8. Solve by Cardan's method : $x^3 - 6x^2 - 6x - 7 = 0$. [5]

Answer any five questions from Q. No. 9 to Q. No. 16 :

9. Let $\sum_{n=1}^{\infty} a_n$ be a series of monotone decreasing positive numbers. Prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges. [5]

10. Test the convergence of the following series :

    a) $1+\frac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^2+\ldots; x>0$

b) $1-\frac{1}{4.3}+\frac{1}{4^2.5}-\frac{1}{4^3.7}+...$ [3+2]

11. a) Find an open cover of [1,2) that does not have a finite subcover. Justify your answer.

b) If A and B are respectively closed and compact subsets of $\mathbb{R}$, then show that $A\cap B$ is compact. [3+2]

12. Let $f:[a,b]\to\mathbb{R}$ be continuous in the closed internal [a, b] and f(a)·f(b) < 0. Prove that there exists at least one point c in the open interval (a, b) such that f(c) = 0. [5]

13. a) If f and g be continuous on the interval [a, b] such that f(x) = g(x) for all rational values x in [a, b], then show that f(x) = g(x) for all $x\in[a,b]$.

b) Let $f(x)=x^2, 0\le x<1$

$= \sqrt{x},\ x\ge 1.$

Prove that f is uniformly continuous on $[0,\infty)$. [3+2]

14. Let $f:[a,b]\to R$ be differentiable on [a,b] and $f'(a)<f'(b)$. Prove that $f'(x)$ assumes every value between $f'(a)$ and $f'(b)$. [5]

15. a) Use mean value theorem to prove that $0<\frac{1}{\log(1+x)}-\frac{1}{x}<1$ for x > 0

b) For each of the following statements cite a function that satisfies :

i) f is differentiable at a point but has no local extremum at that point.

ii) f has no derivative at a point but has a local extremum at that point.

iii) f has a derivative at a point and has a local extremum at that point. [2+3]

16. a) Assume that f has a finite derivative in (a, b) and is continuous on [a, b] with f(a) = f(b) = 0. Prove that for all $K\in R$, there is $c\in(a,b)$ such that $f'(c)=Kf(c)$

b) Evaluate : $\underset{x\to 0+}{\mathrm{Lt}}\ x^x$ [3+2]

Group – B

Answer any four from Question No. 17-22 and any three from Question No. 23-27.

17. Let A be a matrix such that (I + A) is nonsingular. Show that A is skew symmetric iff $B = (I-A)(I+A)^{-1}$ is orthogonal. [3+2]

18. Prove by Laplace method :

$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix} = (af-be+cd)^2$$ [5]

19. Determine the Rank of the matrix reducing it to row-reduced echelon form.

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & 2 & 1 & 5 \\ 1 & 3 & -1 & 0 \\ 1 & 7 & -4 & 1 \end{bmatrix}$$ [4+1]

20. Define linear sum of two subspaces. If U and W be two subspaces of a vector space V over a field F, prove that their linear sum U + W is again a subspace of V. [1+4]

21. Prove that the set {(1, 2, 3), (2, 3, 0), (3, 0, 1)} is a basis of R^3. Show that (3, 1, 7) can replace (3, 0, 1) to form a new basis. [3+2]

22. If $\{\alpha_1, \alpha_2, ...\alpha_n\}$ be a basis of a vector space V over a field F and a non-zero vector β of v is expressed as $\beta = c_1\alpha_1 + c_2\alpha_2 + ... + c_n\alpha_n, c_i \in F$
Then if $c_j \neq 0$, prove that $\{\alpha_1\alpha_2, ...\alpha_{j-1}, \beta, \alpha_{j+1}, ..., \alpha_n\}$ is a new basis of V. [5]

23. a) A hospital has the following requirements for nurses :

Period	Clock Time (24 Hours day)	Minimum number of nurses required
1	6a.m – 10a.m	60
2	10a.m – 2p.m	70
3	2p.m – 6p.m	60
4	6p.m – 10 p.m	50
5	10p.m – 2 a.m	20
6	2a.m – 6a.m	30

Nurses report to the hospital wards at the beginning of each period and work for eight consecutive hours. The hospital wants to determine the minimum number of nurses so that there may be sufficient number of nurses available for each period. Formulate this as an L.P.P. [5]

b) $x_1 = 1, x_2 = 1, x_3 = 1$ and $x_4 = 0$ is a feasible solution of the system of equations
$x_1 + 2x_2 + 4x_3 + x_4 = 7$
$2x_1 - x_2 + 3x_3 - 2x_4 = 4$
Reduce the F.S. to two different B.F.S. [5]

24. a) Solve the L.P.P by Simplex method. [5]
Maximize, $z = 3x_1 - x_2$ subject to
$-x_1 + x_2 \geq 2$
$5x_1 - 2x_2 \geq 2$
$x_1 \geq 0$
$x_2 \geq 0$

b) Using the two phase method, show that a feasible solution does not exist to the following problem :
Min $Z = x_1 + x_2$ subject to the constraints [5]
$3x_1 + 2x_2 \geq 30$
$2x_1 + 3x_2 \geq 30$
$x_1 + x_2 \leq 5$
$x_1 \geq 0,\ x_2 \geq 0$

25. a) Show that an assignment problem is an L.P.P [4]

b) A salesman estimates that the following would be the additional profits obtained on his route by visiting the five cities as shown in the table. The salesman can visit each of the cities once and only once. Determine the optimum sequence the salesman should follow to maximise his additional profit.

From City [6]

To City	**1**	**2**	**3**	**4**	**5**
1	0	10	22	21	18
2	11	0	12	17	19
3	17	11	0	22	24
4	20	19	23	0	24
5	10	22	15	16	0

26. a) The Head of the department has five jobs J_1, J_2, J_3, J_4, J_5 and five subordinates M_1, M_2, M_3, M_4, M_5. The number of hours each man would take to perform each job is as follows :

	M_1	M_2	M_3	M_4	M_5
J_1	1	3	2	3	6
J_2	2	4	3	1	5
J_3	5	6	3	4	6
J_4	3	1	4	2	2
J_5	1	5	6	5	4

How would the jobs be allocated to minimize the total time? [5]

b) Solve the following transportation problem :

	D_1	D_2	D_3	D_4	Supply
O_1	10	20	5	7	10
O_2	13	9	12	8	20
O_3	4	5	7	9	30
O_4	14	7	1	0	40
O_5	3	12	5	19	50
Demand	60	60	20	10	

27. a) Show that any point of a convex polyhedron can be expressed as a convex combination of its extreme points. [5]

b) Prove that if any of the constraints in the primal problem be a perfect equality, then the corresponding dual variable is unrestricted in sign. [5]